Enumerating the ways of distributing n balls into k boxes

devblogs.microsoft.com/oldnewthing/20141027-00

October 27, 2014

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Suppose you had n indistinguishable balls and k distinguishable boxes. Enumerate the ways of distributing the balls into boxes. Some boxes may be empty.

We can represent each distribution in the form of n stars and $k - 1$ vertical lines. The stars represent balls, and the vertical lines divide the balls into boxes. For example, here are the possible distributions for $n = 3$, $k = 3$:

***||	3+0+0
**|*|	2+1+0
**||*	2+0+1
*|**|	1+2+0
||*	1+1+1
*||**	1+0+2
|***|	0+3+0
|**|*	0+2+1
|*|**	0+1+2
||***	0+0+3

This visualization is known in combinatorics circles as stars and bars.

From this visualization, we see that what we are doing is taking n + k − 1 slots, and in each slot placing a star or a bar, subject to the constraint that there be n stars and $k - 1$ bars. Another way of looking at this is that we are choosing a subset of size $k - 1$ from a set of size $n + k - 1$ (the subset specifying where the bars go).

Now we can fire up our subset-generating machine.

```
function Distributions(n, k, f) {
 Subsets(n + k - 1, k - 1, function(s) {
  s.push(n + k);
  f(s.map(function(v, i) { return v - (s[i-1]||0) - 1; }));
  s.pop();
 });
}
```

We ask to generate subsets of size $k - 1$ from a set of size $n + k - 1$. For each such subset, we draw an artificial bar at the end (slot $n + k$), then calculate the number of stars between the bars. The number of stars between two bars is the distance between the two bars, minus 1 because the bar takes up space, too.

Another solution is to reduce this to a problem we already know how to solve: enumerating integer compositions. After distributing the balls into boxes, we go around like Santa Claus and give each box one extra ball, which produces a composition. Conversely, for any composition, remove one ball from each box, and you get a distribution.

```
function Distributions(n, k, f)
{
 Compositions(n + k, k, function(s) {
  f(s.map(function(v) { return v - 1; }));
 });
}
```

We added k extra balls, so we need to generate compositions of $n + k$. When we get each composition, we take one ball away from each box and call that the distribution.

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